WHAT IS THE THEORY ZFC WITHOUT POWER SET?

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Abstract. We show that the theory ZFC-, consisting of the usual axioms of ZFC but with the power set axiom removed—specifically axiomatized by extensionality, foundation, pairing, union, infinity, separation, replacement and the assertion that every set can be well-ordered—is weaker than commonly supposed and is inadequate to establish several basic facts often desired in its context. For example, there are models of ZFC- in which ω_1 is singular, in which every set of reals is countable, yet ω_1 exists, in which there are sets of reals of every size $\aleph_n$, but none of size $\aleph_\omega$, and therefore, in which the collection axiom fails; there are models of ZFC- for which the Łoś theorem fails, even when the ultrapower is well-founded and the measure exists inside the model; there are models of ZFC- for which the Gaifman theorem fails, in that there is an embedding $j : M \to N$ of ZFC- models that is Σ_1-elementary and cofinal, but not elementary; there are elementary embeddings $j : M \to N$ of ZFC- models whose cofinal restriction $j : M \to \bigcup j\,"\, M$ is not elementary. Moreover, the collection of formulas that are provably equivalent in ZFC- to a Σ_1-formula or a Π_1-formula is not closed under bounded quantification. Nevertheless, these deficits of ZFC- are completely repaired by strengthening it to the theory ZFC$^-$, obtained by using collection rather than replacement in the axiomatization above. These results extend prior work of Zarach [Zar96].

1. Introduction

Set theory without the power set axiom is used in arguments and constructions throughout the subject and is usually described simply as having all the axioms of ZFC except for the power set axiom. This theory arises frequently in the large cardinal theory of iterated ultrapowers, for example, and perhaps part of its attraction is an abundance of convenient natural models, including $\langle H_\kappa, \in\rangle$ for any uncountable regular cardinal κ, where H_κ consists of sets with hereditary size less than κ. When prompted, many set theorists offer a precise list of axioms: extensionality, foundation, pairing, union, infinity, separation, replacement and choice. For instance, the theory is described as "ZFC with the Power Set Axiom deleted" by [Kan03] (ch. 19, p. 244) and as "set theory without the Power Set Axiom" by [Jec03] (ch. 19, p. 354) in their treatment of iterated ultrapowers, with ZFC and "set theory" referencing the list of axioms given above. Various other authors describe the theory as "ZFC minus the power-set axiom" [Abr10] (ch. 5, p. 380), "standard axioms of ZFC excluding the powerset axiom" [Nee10] (ch. 22, p. 1883), or verify replacement (rather than collection) when determining whether a given

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structure satisfies the theory. The authors of the current paper have in the past described the theory as ZFC without the power set axiom [HJ10, Git11].[1] Let us denote by ZFC- the theory having the axioms listed above with the axiom of choice taken to mean Zermelo's well-ordering principle, which then implies Zorn's Lemma as well as the existence of choice-functions. These alternative formulations of choice are not all equivalent without the power set axiom as is proved by Zarach in [Zar82],[2] who initiated the program of establishing unintuitive consequences of set theory without power set, which we carry on in this paper. Since our aim is to emphasize the weakness of the replacement scheme in the absence of power set, we opt here for the strongest variant of choice.

In this article, we shall prove that this formulation of set theory without the power set axiom is weaker than may be supposed and is inadequate to prove a number of basic facts that are often desired and applied in its context. Specifically, we shall prove that the following behavior can occur with ZFC- models.

1. (Zarach) There are models of ZFC- in which the countable union of countable sets is not necessarily countable, indeed, in which ω_1 is singular, and hence the collection axiom scheme fails.
2. (Zarach) There are models of ZFC- in which every set of reals is countable, yet ω_1 exists.
3. There are models of ZFC- in which for every $n < \omega$, there is a set of reals of size $\aleph_n$, but there is no set of reals of size $\aleph_\omega$.
4. The Łoś ultrapower theorem can fail for ZFC- models.
   (a) There are models $M \models$ ZFC- with an M-normal measure μ on a cardinal κ in M, for which the ultrapower by μ, using functions in M, is well-founded, but the ultrapower map is not elementary.
   (b) Such violations of Łoś can arise even with internal ultrapowers on a measurable cardinal κ, where $P(\kappa)$ exists in M and $\mu \in M$.
   (c) There is $M \models$ ZFC- in which $P(\omega)$ exists in M and there are ultrafilters μ on ω in M, but no such M-ultrapower map is elementary.
5. The Gaifman theorem [Gai74] can fail for ZFC- models.
   (a) There are Σ_1-elementary cofinal maps $j : M \to N$ of transitive ZFC- models, which are not elementary.
   (b) There are elementary maps $j : M \to N$ of transitive ZFC- models, such that the canonical cofinal restriction $j : M \to \bigcup j\text{ " } M$ is not elementary.
6. Seed theory arguments can fail for ZFC- models. There are elementary embeddings $j : M \to N$ of transitive ZFC- models and sets $S \subseteq \bigcup j\text{ " } M$ such that the seed hull $\mathbb{X}_S = \{j(f)(s) \mid s \in [S]^{<\omega}, f \in M\}$ of S is not an elementary submodel of N. In this case, the restriction $j : M \to \mathbb{X}_S$ is a cofinal Σ_1-elementary map that is not elementary.
7. The collection of formulas that are provably equivalent in ZFC- to a Σ_1-formula or a Π_1-formula is not closed under bounded quantification.

[1] In contrast, Zarach in [Zar82] explicitly includes the collection scheme instead of replacement in his definition of ZF^-, as does Devlin in [Dev84] (p. 119). Jensen, also, reportedly used collection rather than replacement in his course notes. The results of this article show that using collection in place of replacement is a critical difference.

[2] Zarach [Zar82] credits Z. Szczpaniak with first showing that there are models of ZF^- in which choice-functions exist but Zermelo's well-ordering principle fails.

The subsequent theorems of this article contain all the details, including additional undesirable behavior for ZFC- models. The counterexample ZFC- models we produce can be arranged to satisfy various natural strengthenings of ZFC-, to include such statements as Hartogs' theorem that $\aleph_\alpha$ exists for every ordinal α, or alternatively, the assertion "*I am* H_{θ^+}," meaning the assertion that θ is the largest cardinal and every set has hereditary size at most θ.

The fact that the replacement axiom scheme does not imply the collection axiom scheme without the power set axiom was first proved by Zarach [Zar96], who knew the situation described in statements (1) and (2), using a general method similar to the one we employ in this article. This article should therefore be viewed as an extension of Zarach's results, particularly to the case of the Łoś and Gaifman theorems, which we find interesting because these theorems are extensively used in the context of set theory without power set.

Although the failure of such properties as we mention above is often thought to revolve around the axiom of choice—there are after all some famous models of ZF + $\neg$AC in which ω_1 is singular and the Łoś theorem fails—our arguments reveal instead a separate issue arising from the inequivalence without the power set axiom of the replacement and collection axiom schemes. In particular, all our counterexample models will satisfy the axiom of choice in any of the usual set formulations, including the existence of choice-functions, Zorn's lemma, and Zermelo's well-ordering principle.

Although the failure of the Łoś and Gaifman theorems for models of ZFC- might seem troubling—these theorems are fundamental in the theory of ultrapowers and iterated ultrapowers, where the use of models of set theory without the power set axiom is pervasive—nevertheless all is well with these applications, for the following reason. Namely, all of the problematic issues identified in this article for the theory ZFC- disappear if one should simply strengthen it to the theory ZFC$^-$, which is axiomatized by the same list of axioms as in the opening of this article where choice is taken to mean that every set can be well-ordered, but the replacement scheme is replaced by the collection scheme.[3] In particular, the reader can readily check that ZFC$^-$ suffices to prove that successor cardinals are regular, and models of ZFC$^-$ satisfy the Łoś theorem and the Gaifman theorem and so on. The somewhat higher minus sign shall serve to remind the reader that the theory ZFC$^-$ is stronger than ZFC-, and it is this stronger version of the theory that holds in all applications of ZFC without power set of which we are aware. For example, if κ is an uncountable regular cardinal, then H_κ is easily seen to satisfy the collection scheme and hence full ZFC$^-$; also, any model of ZFC- with the global choice axiom, in the form of a global class well-ordering of the universe, must also satisfy ZFC$^-$, since the global choice class allows us to transform instances of collection into instances of replacement, by picking the least witness, and thereby satisfy them. Thus, in all the uses of models of set theory without power set of which we are aware, one actually has the stronger theory ZFC$^-$ anyway, and thereby avoids the problematic issues of ZFC- that we identify in this article.

The main point of this paper, therefore, is to reveal what can go wrong when one naively uses ZFC- in a set-theoretic argument for which one should really be

[3]We believe the choice axiom in ZFC$^-$ should refer to Zermelo's well-ordering principle, since in the context of set theory without power set the existence of choice-functions does not suffice to prove that every set can be well-ordered, even if the collection axiom scheme holds (see [Zar82]).

using ZFC$^-$, and to point out that if one indeed would use ZFC$^-$, then all standard arguments carry through as expected. In other words, our point is that ZFC- is the wrong theory, and in almost all applications, set theorists should be using ZFC$^-$ instead.

Before continuing, perhaps it would be helpful for us to point out where in the usual arguments that we have mentioned one uses collection rather than merely replacement. For example, when proving that ω_1 is regular, or more generally when proving that the countable union of countable sets is countable, one has a countable sequence $\langle X_n \mid n<\omega\rangle$ of countable sets X_n. One would like to use the axiom of choice to select witnessing bijections $f_n:\omega\cong X_n$ and then, with a pairing function on ω, use the map $\langle n,k\rangle\mapsto f_n(k)$ to map ω surjectively onto $\bigcup_n X_n$. The problem with carrying out this argument in ZFC- is that in order to apply the axiom of choice in the first place, we would need to have a sequence $\langle F_n \mid n<\omega\rangle$ of nonempty sets F_n consisting of bijections $f:\omega\cong X_n$ to which to apply it. Without the power set axiom, however, we do not know that the collection of all bijections from ω to X_n forms a set, and so we somehow need first to reduce to a set of witnesses before applying AC to choose individual witnesses. Thus, we would seem to want the collection scheme, which would exactly allow us to do that, and so the argument does work in ZFC$^-$. A similar issue arises when proving that successor cardinals κ^+ are regular. The results of this paper show that with only replacement, one simply cannot push the argument through.

A similar issue arises when proving the Łoś theorem in the forward direction of the existential case. One has the ultrapower $j:M\to \mathrm{Ult}(M,\mu)$ and $M\models \exists x\,\varphi(x,f(\alpha))$ for μ-almost every α, where the ultrapower is constructed using functions in M. What one would like to do is to apply the axiom of choice in order to select a witness x_α such that $M\models\varphi(x_\alpha,f(\alpha))$, for each α for which there is such a witness. But in order to apply the axiom of choice here, one must first know that there is a set of witnesses x such that $\varphi(x,f(\alpha))$ from which to choose, which may be accomplished using collection. But without collection, it follows by our results that the argument simply cannot succeed. Note that replacement in M suffices to prove the Łoś theorem for Δ_0-formulas since if there is a function $g\in M$ such that $M\models\exists x{\in}g(\alpha)\,\varphi(x,f(\alpha))$ for μ-almost every α, then the witnesses can be chosen out of $\bigcup\mathrm{ran}(g)$, which is a set by replacement. Since ultrapower embeddings of models of ZFC- are cofinal[4] by replacement, it follows that they are always Σ_1-elementary. We will produce a model of ZFC- with an ultrapower embedding that is not Σ_2-elementary and for which the Łoś theorem fails already for Σ_1-formulas. This will also demonstrate a failure of the Gaifman theorem.

In its full generality, the Gaifman theorem [Gai74] states that if $M\models$ ZFC$^-$ and $j:M\to N$ is a Δ_0-elementary cofinal map, then j is fully elementary.[5] No additional assumptions are made on N, and the models M and N need not be transitive. It is easy to see that N satisfies the pairing axiom, since if x,y are any elements in N, then by cofinality there are $a,b\in M$ such that $x\in j(a)$ and $y\in j(b)$, and so, by replacement in M, there is the cartesian product $C=a\times b$ so that

[4]An embedding $j:M\to N$ is *cofinal* if every $x\in N$ is an element of $j(y)$ for some $y\in M$.

[5]In Gaifman's original formulation, the theorem stated that if M is a model of Zermelo's set theory (ZF axioms with the separation scheme but no replacement scheme), and $j:M\to N$ is a Δ_0-elementary cofinal map, then it is fully elementary. Gaifman, then pointed out that the existence of cartesian products in M suffices in the place of power sets, and thus, in our context, the theorem holds if $M\models$ ZFC$^-$.

$M \models \forall u{\in}a\, \forall v{\in}b\, \exists w{\in}C\, w{=}\langle u, v\rangle$, which is a Δ_0-formula. It follows that, in both M and N, blocks of like quantifiers in formulas may be contracted to a single quantifier through applications of the pairing axiom. The proof now proceeds by induction on the complexity of formulas, and since j is Σ_1-elementary by cofinality, the critical case of the induction occurs when j is assumed to be Σ_n-elementary for some $n > 0$ and $M \models \forall x \in a\, \exists y\, \varphi(x, y, p)$ for some formula $\varphi \in \Pi_{n-1}$ and some sets $a, p \in M$. Using collection in M, there is a set b such that $M \models \forall x{\in}a\, \exists y{\in}b\, \varphi(x, y, p)$, and since $a{\times}b$ exists in M, one may use separation in M to obtain a set C such that M satisfies $\forall x{\in}a \exists y{\in}b\, \langle x, y\rangle \in C$ and $\forall x{\in}a \forall y{\in}b\, \big[\langle x, y\rangle \in C \Rightarrow \varphi(x, y, p)\big]$. The first statement is Σ_0, and the second is Π_{n-1}, so that both statements transfer to N by the inductive hypothesis. It follows that $N \models \forall x{\in}j(a)\, \exists y{\in}j(b)\, \varphi(x, y, j(p))$, completing the proof of the critical case.[6] We shall show that the use of collection in M is essential to this argument by producing a cofinal Δ_0-elementary map $j : M \to N$ of ZFC- models that is not elementary.

An analogous issue as in the Łoś theorem arises when proving that for a given elementary embedding $j : M \to N$ and some $S \subseteq \bigcup j\,``\,M$ the *seed hull* of S via j, meaning the structure $\mathbb{X}_S = \{j(f)(s) \mid s \in [S]^{<\omega}, f \in M\}$, is an elementary submodel of N. This is usually shown by verifying the Tarski-Vaught test for $\mathbb{X}_S \subseteq N$, and so one has that $N \models \exists y\, \varphi(y, j(f)(s))$ where $s \in j(D)$ for some $D \in M$. One would like to use a Skolem function $g \in M$ for $\varphi(y, f(x))$, meaning that for each $x \in D$, one lets $g(x)$ be any witness y such that $\varphi(y, f(x))$ holds in M, if such a witness exists. But, the usual proof that such g exists in M uses collection in M, since one can appeal to AC in M to choose witnesses only after sufficiently many such witnesses have been collected to a set in M. This shows that $\mathbb{X}_S \prec N$ in the case when $M \models \mathrm{ZFC}^-$, but in our context, when $M \models$ ZFC-, we will find elementary embeddings $j : M \to N$ and sets S of seeds such that $\mathbb{X}_S \not\prec N$. Note that replacement in M suffices to see that $\mathbb{X}_S$ is Δ_0-elementary in N, since in this case the existential quantifier in the Tarski-Vaught test may be bounded by some $j(h)(s)$ and therefore the witnesses for the Skolem function $g \in M$ can be chosen out of $\bigcup \mathrm{ran}(h)$. The map $j : M \to \mathbb{X}_S$ is thus cofinal and hence Σ_1-elementary, but not fully elementary, and so is the map $\pi \circ j : M \to \mathrm{ran}(\pi)$, where π denotes the Mostowski collapse of $\mathbb{X}_S$, which means that both maps demonstrate a failure of the Gaifman theorem. Moreover, our maps will allow for seed hulls $\mathbb{X}_S$ that are not generated by a single seed, so that we will get these failures for embeddings that are not ultrapower maps.

2. Badly behaved models of ZFC-

Let us now begin our project by describing a variety of models of ZFC-, in each case making observations about how that model reveals a compatibility of various undesirable behaviors with the theory ZFC-. Although there is some redundancy in this bad behavior of the models in that several different example models reveal the same deficiency in ZFC-, we have nevertheless included all the examples to illustrate the range and flexibility of the constructions. We shall organize our presentation principally around the construction methods, rather than around the various undesirable behavior for models of ZFC-.

[6]The proof of Gaifman's result becomes a much easier induction on the complexity of formulas, if one assumes at the outset that N is a model of ZFC^-, and this folklore version of the theorem was already known before Gaifman's result according to [Gai74].

2.1. **The Lévy collapse of $\aleph_\omega$.** Let $\mathrm{Coll}(\omega,{<}\aleph_\omega)$ be the Lévy collapse of $\aleph_\omega$, meaning the finite-support product $\Pi_n \mathrm{Coll}(\omega,\aleph_n)$, and suppose that $G\subseteq\mathrm{Coll}(\omega,{<}\aleph_\omega)$ is V-generic. The cardinal $\aleph_\omega$ is collapsed to ω in $V[G]$, but it remains a cardinal in every model $V[G_n]$, where $G_n=G\cap\mathbb{P}_n$ is the restriction of G to the collapse forcing $\mathbb{P}_n=\Pi_{k\leq n}\mathrm{Coll}(\omega,\aleph_k)$ which proceeds only up to $\aleph_n$. Our desired model is

$$W=\bigcup_{n<\omega}V[G_n],$$

which is the union of the forcing extensions $V[G_n]$ arising during initial segments of the collapse. The model W is a definable class in $V[G]$ using a parameter from V determined by $\mathrm{Coll}(\omega,{<}\aleph_\omega)$ to define the ground model V (by a result of Laver's [Lav07]) and the filter G. Although the definability of W will not be required for this argument, it will be used in the later constructions. Note that W depends not only on $V[G]$, but also on the way that G is presented, and there are automorphisms of the forcing that do not preserve W.[7] Nevertheless, any automorphism of any particular $\mathbb{P}_n$ induces an automorphism of $\mathbb{P}$ that preserves W. Note that every stage m of the forcing has for any two conditions with the same domain, an automorphism generated by mapping one to the other, and combining these automorphisms for $m\leq n$ produces automorphisms of $\mathbb{P}_n$ that will be used later in the argument.

Let us now verify that $W\models\mathrm{ZFC}^-$, by checking each axiom in turn. Most of them follow easily. For example, extensionality and foundation hold in W, because it uses the $\in$-relation of $V[G]$ and is transitive; union and pairing hold since W is the union of a chain of models of ZFC, which each have the required union and pairing sets; infinity holds since $\omega\in V\subseteq W$; and Zermelo's well-ordering principle holds in W, because every set in W is in some $V[G_n]$, where it has a well-order that survives into W; and separation follows from replacement, and so needn't be considered separately.

So it remains to verify only the replacement axiom scheme in W. Suppose that A and z are sets in W, and that $W\models\forall a{\in}A\,\exists!b\,\varphi(a,b,z)$, where the exclamation point expresses that there is a unique such b. We may assume without loss of generality that $A,z\in V$, since otherwise we have $A,z\in V[G_n]$ for some n and we may replace V with $\overline{V}=V[G_n]$, using the fact that forcing with the tail of the product gives rise to the same W. We claim next that for any $a\in A$, the unique witness b for a is also in V. Fix any $a\in A$, and let b be the unique witness such that $W\models\varphi(a,b,z)$. It must be that $b\in V[G_n]$ for some n, and so $b=\dot b_G$ for some $\mathbb{P}_n$-name $\dot b$. Note that we specifically chose a $\mathbb{P}_n$-name to ensure that its interpretation by any generic $\overline{G}$ will end up in $V[\overline{G}_n]$, a circumstance crucial to the later part of the argument. Suppose that some condition $p\in G$ forces $\varphi^{\dot W}(\check a,\dot b,\check z)$, where $\dot W$ is the forcing language definition of the class W using the canonical name $\dot G$ for the generic filter. Note that the stage $n+1$ forcing $\mathrm{Coll}(\omega,\aleph_{n+1})$ is isomorphic to the product $\mathbb{P}_n\times\mathrm{Coll}(\omega,\aleph_{n+1})$, and so we may view the stage $n+1$ forcing as first adding another mutually generic filter $\overline{G}_n\subseteq\mathbb{P}_n$ and then performing the rest of the

[7] For example, one could view each stage of the forcing as also adding a Cohen generic real via an isomorphism of $\mathbb{P}_n$ with the product of $\mathbb{P}_n$ and the Cohen poset, and then consider an automorphism in which the sequence of these reals is added before the first collapsing stage via an isomorphism of $\mathrm{Coll}(\omega,{<}\aleph_\omega)$ with the countable product of Cohen posets and $\mathrm{Coll}(\omega,{<}\aleph_\omega)$, so that for one generic filter, the sequence is not in W, but for the automorphic copy of the generic filter, it is.

forcing. That is, $\mathbb{P}$ is isomorphic to the forcing that forces with two copies of $\mathbb{P}_n$, first using the actual copy and then using an additional copy, before the product proceeds with stage $n+1$ and the rest. By swapping these two copies of $\mathbb{P}_n$ inside $\mathbb{P}_{n+1}$, we produce a slightly different V-generic filter $\overline{G}$, such that G_n and $\overline{G}_n$ are mutually generic for $\mathbb{P}_n$, but $V[G] = V[\overline{G}]$. Since we are rearranging the filter only within $\mathbb{P}_{n+1}$, it follows that $\dot{W}_G = \dot{W}_{\overline{G}}$. We may furthermore assume without loss of generality that $p \in \overline{G}$, by applying if necessary an additional automorphism (as described above) to the second copy of $\mathbb{P}_n$ that we constructed in $\mathbb{P}_{n+1}$, before we perform the swap of the two copies of $\mathbb{P}_n$. It follows that $\varphi^W(a, \dot{b}_{\overline{G}}, z)$, and so by the uniqueness of b, we have $b = \dot{b}_G = \dot{b}_{\overline{G}}$. This equation implies that b is in $V[G_n] \cap V[\overline{G}_n]$, which is equal to V since these filters are mutually generic, by a result of Solovay [Sol70]. Thus, $b \in V$ as we claimed. Now, for any given $a \in A$ the question of whether a given $b \in V$ has $\varphi^W(a, b, z)$ must be decided by $\mathbb{1}$, since $\mathbb{P}$ is almost homogeneous by automorphisms (as described above) affecting only finitely many coordinates and such automorphisms do not affect the value of $\dot{W}$.[8] Thus, the set $\{\, b \mid \exists a{\in}A\, \varphi^W(a, b, z) \,\}$ exists in V by replacement in V, and hence also exists in W as desired. This completes the argument that W is a model of replacement, and so $W \models$ ZFC−.

We shall now make further observations about this model W in order to prove that ZFC− is consistent with various undesirable behaviors. The first few observations were made already by Zarach [Zar96], using a method fundamentally similar to ours. In the subsequent sections, we shall modify this construction in order to obtain failures of the Łoś and Gaifman theorems, among others.

Theorem 1 (Zarach)**.** *It is consistent with* ZFC− *that ω_1 exists but is singular and hence that a countable union of countable sets can be uncountable.*

Proof. Consider the model W as constructed in the Lévy collapse of $\aleph_\omega$ above. Note that every cardinal $\aleph_n^V$ is collapsed in $V[G_n]$ and hence in W, but $\aleph_\omega^V$ remains a cardinal in every $V[G_n]$ and hence also in W. Thus, $\omega_1^W = \aleph_\omega^V$, which has cofinality ω in V and hence in W, as witnessed by the sequence $\langle \aleph_n \mid n < \omega \rangle \in V \subseteq W$. So W satisfies that ω_1 is singular. In particular, W satisfies that ω_1 is a countable union of countable sets, as $\omega_1^W = \bigcup\{\, \aleph_n^V \mid n \in \omega \,\}$. $\square$

Theorem 2 (Zarach)**.** *It is consistent with* ZFC− *that the collection scheme fails. Hence, replacement and collection are not equivalent without the power set axiom.*

Proof. The collection scheme fails in the model W above because for each countable ordinal α in W there is a surjective function $f : \omega \twoheadrightarrow \alpha$, but there is no set B in W collecting a family of such functions, since every B in W arises in some $V[G_n]$ and therefore contains no functions collapsing the ordinals α in the interval $[\aleph_{n+1}^V, \aleph_\omega^V)$. $\square$

In the proof above, we could have also obtained a violation of collection using domain ω instead of ω_1 by considering the sequence $\langle \aleph_n^V \mid n < \omega \rangle$, which is an element of V and hence of W, and observing that W has bijections $f : \omega \twoheadrightarrow \aleph_n$, but a family of such functions cannot be collected.

[8] A forcing $\mathbb{P}$ is *almost homogeneous* if for any two conditions p and q there is an automorphism π of $\mathbb{P}$ such that $\pi(p)$ is compatible with q. Since $\pi([\![\, \varphi(\tau) \,]\!]) = [\![\, \varphi(\tau^\pi) \,]\!]$, where τ^π is the name produced by hereditary application of π, it follows for such forcing that the Boolean value of any statement whose parameters are not affected by π is either 0 or $\mathbb{1}$.

Theorem 3 (Zarach). *It is consistent with* ZFC- *that every set of reals is countable, yet* ω_1 *exists.*

Proof. Consider the model W constructed as above but starting with a ground model V in which $2^{<\aleph_\omega} = \aleph_\omega$. Every set of reals in such a W is in $V[G_n]$ for some n, and the reals of this model have size $\aleph_m$ for some m by a nice name counting argument and the assumption $2^{<\aleph_\omega} = \aleph_\omega$. Thus, the reals of every $V[G_n]$ become countable in some further $V[G_m]$ and hence in W, so every set of reals in W is countable in W. But W satisfies that ω_1 exists and indeed $\aleph_\alpha$ exists for every ordinal α, since above $\aleph_\omega$ the cardinals of W agree with the cardinals of V. □

The construction is general, and it adapts to forcing with the Lévy collapse of $\aleph_\kappa$ to κ, meaning the bounded-support product $\mathrm{Coll}(\kappa, {<}\aleph_\kappa) = \Pi_{\beta<\kappa} \mathrm{Coll}(\kappa, \aleph_\beta)$. If $G_\gamma = G \cap \mathbb{P}_\gamma$, where $\mathbb{P}_\gamma = \Pi_{\beta\leq\gamma} \mathrm{Coll}(\kappa, \aleph_\beta)$ is the forcing up to γ, then let $W = \bigcup_{\gamma<\kappa} V[G_\gamma]$. As long as κ is a regular cardinal with $\aleph_\beta^{<\kappa} < \aleph_\kappa$ for all $\beta < \kappa$, it follows by essentially the same arguments as before that W satisfies ZFC- but not collection, and that the cardinal κ^+ exists in W but has cofinality κ there. Moreover, the model W is closed in $V[G]$ under sequences of length less than κ, so this construction provides a general method for obtaining badly behaved inner models $W \models$ ZFC- that are as closed as desired.

As we did in this section, we will start many arguments in this article with a model V of ZFC and produce a model of the desired theory with ZFC- by finding an inner model of a forcing extension of V. By the usual forcing methods, such as taking a quotient of a Boolean-valued model, these arguments show that if ZFC is consistent, then so is ZFC- with the stated extra properties. One can omit the need for this consistency assumption and prove in ZFC alone that there are transitive models of ZFC- with the desired properties, simply by forcing over a suitably large H_{θ^+} rather than V itself. For example, one could first make a countable elementary submodel of such an H_{θ^+}, and then build the generic extension by meeting the countably many dense sets. We carry out several such arguments in section §2.7. In any case we find the models W obtained by forcing over models of ZFC to be more striking, since they have V as an inner model, and for example also satisfy Hartogs' theorem that for every ordinal α, the cardinal $\aleph_\alpha$ exists, and many other attractive properties.

2.2. **Pumping up the continuum.** In the next example, we shall force to pump up the continuum instead of forcing to collapse cardinals as in the previous example. Let us start in a model V of ZFC in which $2^\omega < \aleph_\omega$, and let $G \subseteq \mathrm{Add}(\omega, \aleph_\omega)$ be V-generic for the forcing to add $\aleph_\omega$-many Cohen reals. Let $W = \bigcup_{n<\omega} V[G_n]$, where $G_n = G \cap \mathrm{Add}(\omega, \aleph_n)$ is the initial segment of the forcing adding only $\aleph_n$-many Cohen reals. We argue that $W \models$ ZFC-. Once again, most of the axioms are easy to verify, since W is the union of an increasing chain of transitive ZFC models $V[G_n]$. It is non-trivial to verify only replacement, which we handle by a similar argument as above. Namely, suppose that $A, z \in W$ and $W \models \forall a{\in}A\, \exists! b\, \varphi(a, b, z)$. As before, we assume that $A, z \in V$, since otherwise they are in some $V[G_k]$, which we may regard as the new ground model. For each $a \in A$, we claim again that the witness b for which $\varphi^W(a, b, z)$ must be in V. In any case $b \in V[G_n]$ for some n, and so there is some $\mathbb{P}_n$-name $\dot{b}$ with $b = \dot{b}_G$ and a condition $p \in G$ with $p \Vdash \varphi^{\dot{W}}(\check{a}, \dot{b}, \check{z})$. We may view the forcing to add G as consisting of first adding $\aleph_n$-many Cohen reals, and then adding $\aleph_n$-many more Cohen reals, and then adding the rest of them,

since this is an isomorphic presentation of the forcing. By using the automorphism that swaps these two mutually generic blocks of $\aleph_n$-many Cohen reals, we may rearrange the filter G to construct another filter $\overline{G}$ for which $V[G] = V[\overline{G}]$, but $V[G_n]$ and $V[\overline{G}_n]$ are mutually generic extensions by $\mathrm{Add}(\omega, \aleph_n)$, while still having $\dot{W}_G = \dot{W}_{\overline{G}}$. As before, we may also apply an additional automorphism to the second copy of $\mathrm{Add}(\omega, \aleph_n)$ if necessary when constructing $\overline{G}$ and assume without loss of generality that $p \in \overline{G}$. It follows that $W \models \varphi(a, \dot{b}_{\overline{G}}, z)$, and consequently, by the uniqueness of b, that $b = \dot{b}_G = \dot{b}_{\overline{G}}$. Thus, b lies in both $V[G_n]$ and $V[\overline{G}_n]$, but the intersection of these models is V by mutual genericity, and so $b \in V$, as we claimed. To complete the argument, observe that since $\mathbb{P}$ is almost homogeneous by automorphisms not affecting $\dot{W}$, we have that for any given $a \in A$ the question of whether a given $b \in V$ has $\varphi^W(a, b, z)$ must be decided by $\mathbb{1}$. So we may in V build the set $\{\, b \mid \exists a{\in}A\, \varphi^W(a, b, z) \,\}$ by using replacement in V. So this set exists in W, and we have verified the replacement scheme in W.

Theorem 4. *It is consistent with* ZFC- *that $\aleph_\omega$ exists and for each n there are sets of reals of size $\aleph_n$, but no set of reals of size $\aleph_\omega$. In particular, there is no set of reals of largest cardinality, and this violates the collection scheme.*

Proof. Consider the model W we just constructed and recall that we assumed $2^\omega < \aleph_\omega$ in the ground model V. Since the forcing $\mathrm{Add}(\omega, \aleph_\omega)$ has the countable chain condition, it preserves all cardinals to $V[G]$ and hence also to W. The reals of $V[G_n]$ exist in W and have size at least $\aleph_n$ there; hence W has sets of reals of size exactly $\aleph_n$ by Zermelo's well-ordering principle. But every set of reals in W is in $V[G_m]$ for some $m < \omega$ and hence must have size less than $\aleph_\omega$. □

As in the previous section, the construction here is also quite general, and it adapts easily in order to produce models W of ZFC- that violate collection but are highly closed in the overall forcing extension $V[G]$. For example, if κ is any regular cardinal with $2^\omega < \aleph_\kappa$ and $G \subseteq \mathrm{Add}(\omega, \aleph_\kappa)$ is V-generic for the forcing to add $\aleph_\kappa$-many Cohen reals, then let $W = \bigcup_{\gamma<\kappa} V[G_\gamma]$, where $G_\gamma = G \cap \mathrm{Add}(\omega, \aleph_\gamma)$. It follows essentially by the same arguments as before that W satisfies ZFC- but not collection, and that W has sets of reals of size $\aleph_\gamma$ for each $\gamma < \kappa$, but no set of reals of size $\aleph_\kappa$. Moreover, the model W is closed under sequences of length less than κ in $V[G]$. Other generalizations are possible also, and we could have instead added subsets to ω_1 or ω_2 or to any cardinal κ, and made similar conclusions.

Let us now consider a natural strengthening of both collection and replacement. The *reflection principle* scheme is the assertion of any formula φ, that for any given set x there is a transitive set A containing x as a subset such that φ is absolute between A and the universe. The reflection principle scheme is provable in ZFC by appealing to the von Neumann V_α hierarchy, which does not exist when the power set axiom fails. The reflection principle scheme implies the collection axiom scheme directly, and in the presence of separation, it also implies the replacement axiom scheme. Failure of collection therefore implies the failure of the reflection principle scheme, as in the previously constructed models W of ZFC-. It is an interesting open question whether the reflection principle scheme is provable in ZFC^-. We suspect, just as Zarach does in [Zar96], that it is not.

The reflection principle scheme is, over ZFC^-, equivalent to a version of the axiom of choice that we call the *dependent choice scheme*, which is the natural class version of Tarski's principle of dependent choices for definable relations. Specifically,

the DC-scheme asserts of any formula φ, that for any parameter z, if for every x there is y with $\varphi(x, y, z)$, then there is an ω-sequence $\langle x_n \mid n < \omega\rangle$ such that $\forall n\, \varphi(x_n, x_{n+1}, z)$. In other words, if φ defines a relation having no terminal nodes, then we can make ω-many dependent choices to find an ω-path through this relation. The reflection principle scheme implies the DC-scheme, since it reduces instances of the DC-scheme to set instances of DC, which then follow from AC. Conversely, to obtain reflection for a particular formula φ from the DC-scheme, we first use collection to see that any given set x can be extended to a transitive set containing existential witnesses for all subformulas of φ with parameters from x, and then we use the DC-scheme to chose an ω-path of such extensions. The next theorem shows that the use of collection when proving this converse direction is essential.

Theorem 5 (Zarach)**.** *It is consistent with* ZFC- *that the* DC*-scheme holds, but the reflection principle scheme fails.*

Proof. Suppose that $2^\omega < \aleph_{\omega_1}$ and that $G \subseteq \mathrm{Add}(\omega, \aleph_{\omega_1})$ is the forcing to add $\aleph_{\omega_1}$-many Cohen reals. If $W = \bigcup_{\gamma<\omega_1} V[G_\gamma]$ where $G_\gamma = G \cap \mathrm{Add}(\omega, \aleph_\gamma)$, then as we discussed earlier in this section, W satisfies ZFC- but not collection and is closed under countable sequences in $V[G]$. The DC-scheme holds in W, since it holds in $V[G]$ and W is a definable class with $W^\omega \subseteq W$ in $V[G]$. However, the reflection principle scheme fails in W, since collection does. □

For any infinite cardinal κ, Lévy [Lév64] introduced the principle DC_κ, which is the assertion that for any set A and any binary set relation R, if for each sequence $\vec{s} \in A^{<\kappa}$ there is a $y \in A$ such that $\vec{s}$ is R-related to y, then there is a κ-sequence $\langle x_\xi \mid \xi < \kappa\rangle$ such that for each $\alpha < \kappa$ the initial sequence $\langle x_\xi \mid \xi < \alpha\rangle$ is R-related to x_α. This natural generalization of Tarski's principle of dependent choices allows for κ-many dependent choices, rather than just ω-many, and it is easy to see that DC_ω is equivalent to the usual principle of dependent choices. Jech showed in [Jec66] that it is relatively consistent with ZF that DC_α holds for all α below any given regular κ but DC_κ fails.

In our context, namely in the theory ZFC- where every set can be well-ordered, we consider the natural class version of DC_κ, namely the principle that we call the DC_κ*-scheme*, which asserts of any formula φ that for any parameter z, if for every x there is y with $\varphi(x, y, z)$, then there is a function f with domain κ such that $\forall \xi{<}\kappa\, \varphi(f\restriction\xi, f(\xi), z)$. It is easy to see that the DC_ω-scheme is equivalent over ZFC- to the DC-scheme. The reflection principle scheme implies the DC_κ-scheme for all cardinals κ, since it reduces it to set instances of DC_κ, which then follow from AC. But, as the next theorem shows, under ZFC- the DC_κ-scheme is not strong enough to prove the reflection principle scheme, or DC_λ if $\lambda > \kappa$.

Theorem 6. *Suppose that κ is any regular cardinal with $2^\omega < \aleph_\kappa$ and that $G \subseteq \mathrm{Add}(\omega, \aleph_\kappa)$ is V-generic. If $W = \bigcup_{\gamma<\kappa} V[G_\gamma]$ where $G_\gamma = G \cap \mathrm{Add}(\omega, \aleph_\gamma)$, then $W \models$ ZFC- has the same cardinals as V and the DC_α-scheme holds in W for all $\alpha < \kappa$, but the DC_κ-scheme and the reflection principle scheme fail. In particular, it is consistent with* ZFC- *that the* DC*-scheme fails.*

Proof. As we discussed earlier in this section, W is a model of ZFC- but not collection, it has sets of reals of size $\aleph_\gamma$ for each $\gamma < \kappa$, but no set of reals of size $\aleph_\kappa$, and $W^{<\kappa} \subseteq W$. Using the chain condition, W and V have the same cardinals. Thus, for each $\alpha < \kappa$, the DC_α-scheme holds in W, but the reflection principle

scheme fails. To see that W does not satisfy the DC_κ-scheme, let $\varphi(x,y)$ assert that y is an infinite set of reals, and if x is any sequence of sets of reals, then $|\bigcup x| < |y|$. Because there are increasingly large sets of reals in W, it follows that for each sequence $\vec{s} \in W^{<\kappa}$ of sets of reals there is a $y \in W$ such that $\varphi(\vec{s}, y)$ holds in W. But there is no κ-path through this relation in W, because the union of any such sequence would be a set of reals in W of size at least $\aleph_\kappa$, a contradiction. Lastly note that if $\kappa = \omega$, the DC-scheme fails. □

Zarach showed already in [Zar96] that there is a model of ZFC- in which the DC-scheme fails. He also showed that over ZFC-, the scheme $\forall\alpha\ \mathrm{DC}_\alpha$ implies the collection scheme and therefore, if collection fails then there must be a cardinal κ such that DC_κ fails as well.

2.3. **The Lévy collapse of an inaccessible cardinal.** Suppose now that κ is an inaccessible cardinal and that G is V-generic for the Lévy collapse of κ, meaning the finite-support product $\mathrm{Coll}(\omega, {<}\kappa) = \Pi_{\beta<\kappa} \mathrm{Coll}(\omega, \aleph_\beta)$. Let $W = \bigcup_{\gamma<\kappa} V[G_\gamma]$, where $G_\gamma = G \cap \mathbb{P}_\gamma$, where $\mathbb{P}_\gamma = \Pi_{\beta\leq\gamma} \mathrm{Coll}(\omega, \aleph_\beta)$ is the forcing up to $\aleph_\gamma$. Note that $W^{<\kappa} \subseteq W$. The argument that $W \models$ ZFC- is identical to those previously given and relies on the fact that $\mathrm{Coll}(\omega, \aleph_{\gamma+1})$ may be viewed via forcing equivalence as first performing another copy of $\mathbb{P}_\gamma = \Pi_{\beta\leq\gamma} \mathrm{Coll}(\omega, \aleph_\beta)$, and then performing the rest of the collapse.

The model W possesses another interesting feature. Note that the filter G_γ is coded by a real in $V[G]$, and every real of $V[G]$ appears in some $V[G_\gamma]$. So an equivalent description of W in $V[G]$ is as the union $W = \bigcup_{r\in\mathbb{R}} V[r]$ (where $V[r]$ denotes the closure of V and r under the Gödel operations).

Theorem 7. *Relative to an inaccessible cardinal, it is consistent with* ZFC- *that ω_1 exists and is regular, but every set of reals is countable, which implies that the collection scheme fails.*

Proof. Consider the model W as constructed just previously. The cardinal κ becomes ω_1 in $V[G]$ and hence also in W, and remains regular there. Every set of reals in W is in $V[G_\gamma]$ for some $\gamma < \kappa$, and becomes countable at a later stage and hence countable in W. This implies that the collection scheme fails in W, because for each countable ordinal α, there is a function $f : \omega \cong \alpha$, but there is no set containing such functions for every α, since from such a set we could construct in W an uncountable set of reals. □

To see the failure of collection in the proof of theorem 7, it was crucial that every set of reals was countable. Indeed, one of Zarach's results on unintuitive consequences of set theory without power set shows that it is relatively consistent with ZFC^- that Hartogs' theorem holds, but there are unboundedly many cardinals whose powersets are proper classes with all subsets of a certain bounded size. For example, he provides in [Zar82] a model of ZFC^- for which $\aleph_\alpha$ exists for each ordinal α, where $P(\omega)$ is a proper class, but every set of reals has size at most ω_1.

Note also that if $W \models$ ZFC- is a model of Hartogs' theorem in which ω_1 exists and is regular and every set of reals is countable, then ω_1 is inaccessible in $L^W \models$ ZFC, and indeed, is inaccessible to reals, for otherwise we would find an uncountable set of reals in some $L[x]^W$, which would remain uncountable in W. So the use of the inaccessible cardinal is necessary for any construction that obtains W as above.

2.4. **The Lévy collapse of a measurable cardinal.** Let us now turn to a version of the construction providing a violation of the Łoś theorem. Namely, in theorem 8 we show that in the Lévy collapse $V[G]$ of a measurable cardinal κ, the inner model W as constructed in section §2.3 has a definable ultrafilter μ^* on $\kappa = \omega_1^{V[G]}$, whose ultrapower is well-founded, but the ultrapower map is not elementary. Thus, it is relatively consistent with ZFC- that this version of the Łoś theorem fails for ultrapowers, where the ultrafilter is fully amenable to the model and indeed definable over the model and the ultrapower is well-founded. Furthermore, we show that the Gaifman theorem fails for this ultrapower embedding, since it is Σ_1-elementary and cofinal, but not fully elementary.

Theorem 8. *If $V[G]$ is the forcing extension by the Lévy collapse $G \subseteq \mathrm{Coll}(\omega, {<}\kappa)$ of a measurable cardinal κ and $W = \bigcup_{\gamma<\kappa} V[G_\gamma]$, where $G_\gamma = G \cap \mathbb{P}_\gamma$ and $\mathbb{P}_\gamma = \Pi_{\beta\leq\gamma} \mathrm{Coll}(\omega, \aleph_\beta)$, then:*

1. $W \models$ ZFC-.
2. *$W^{<\kappa} \subseteq W$ in $V[G]$.*
3. *In W there is a definable W-normal measure μ^* on κ.*
4. *The ultrapower $\overline{M} \cong \mathrm{Ult}(W, \mu^*)$ taken in $V[G]$ using functions on κ in W is well-founded and $\overline{M} \models$ ZFC-.*
5. *The class $\overline{M}$ and the ultrapower map are definable in W.*
6. *The Łoś theorem fails for this ultrapower at the Σ_1-level.*
7. *The Gaifman theorem fails for the ultrapower map $j : W \to \overline{M}$, since it is Σ_1-elementary and cofinal, but not Σ_2-elementary.*

Proof. We already observed in section §2.3 that $W \models$ ZFC- but not collection, and that $W^{<\kappa} \subseteq W$. Let μ be any normal measure on κ in V. Note that every initial segment $\mathbb{P}_\gamma$ of the forcing is small relative to κ. Thus, by the Lévy-Solovay theorem [LS67] it follows that κ remains measurable in $V[G_\gamma]$, and indeed, the filter μ_γ generated by μ in $V[G_\gamma]$ is a normal measure on κ in $V[G_\gamma]$. Let μ^* be the filter on κ generated by μ in W, which is the same as $\bigcup_\gamma \mu_\gamma$. Note that μ^* is not in W, since it is not in any $V[G_\gamma]$, but it is definable over W from parameter μ, since a set is in μ^* if and only if it covers an element of μ. In fact, μ^* is a W-normal measure on κ, since every subset of κ in W is in some $V[G_\gamma]$ and hence is measured by μ_γ, and every regressive function on κ in W is in some $V[G_\gamma]$ and hence is constant on a μ_γ-large set there. Moreover, the measure μ^* is countably complete in $V[G]$ since W contains all its ω-sequences from $V[G]$. We can construct in $V[G]$, since W is definable there, the ultrapower $\mathrm{Ult}(W, \mu^*)$ using functions on κ in W. Since μ^* is countably complete, it follows by the usual argument that $\mathrm{Ult}(W, \mu^*)$ is well-founded, and thus, we can Mostowski collapse it to obtain the ultrapower map $j : W \to \overline{M}$. In fact, we can both construct and collapse the ultrapower in W itself, though this is not immediately obvious as Scott's trick that is crucial to this process may fail in the absence of power sets (see our remark after the proof). Note that the ultrapower map j is Σ_1-elementary and cofinal, by our remarks in the introduction of this article.

Since the forcing $\mathbb{P}_\gamma$ for $\gamma < \kappa$ is small relative to κ, it follows that the ultrapower map $j_0 : V \to M$ by μ in V lifts uniquely to an elementary embedding $j_\gamma : V[G_\gamma] \to M[G_\gamma]$, which necessarily equals the ultrapower map by μ_γ in $V[G_\gamma]$. In fact, whenever $\gamma < \delta$ are ordinals below κ, then $j_\delta \restriction V[G_\gamma] = j_\gamma$, since when we lift the ultrapower j_γ to $V[G_\delta]$ we obtain by the smallness of the corresponding

forcing precisely the ultrapower by μ_δ, which is the same as j_δ. The union $\bigcup_{\gamma<\kappa} j_\gamma$ is thus a well-defined map. We claim that $j = \bigcup_{\gamma<\kappa} j_\gamma$ and $\overline{M} = \bigcup_{\gamma<\kappa} M[G_\gamma]$ via the isomorphism $[f]_{\mu_\gamma} \mapsto [f]_{\mu^*}$ whenever $f : \kappa \to V[G_\gamma]$ is a function in $V[G_\gamma]$. This map is well-defined and $\in$-preserving since $[f]_{\mu_\gamma} = j_\gamma(f)(\kappa) = j_\delta(f)(\kappa) = [f]_{\mu_\delta}$ whenever $\gamma < \delta$ and $[f]_{\mu_\gamma} \in M[G_\gamma]$. The map is clearly onto, and it follows that $[f]_{\mu_\gamma} = [f]_{\mu^*}$ whenever $[f]_{\mu_\gamma} \in M[G_\gamma]$. It follows that $\overline{M} \models$ ZFC-, since it is obtained from M by the Lévy collapse of κ in the manner of section §2.3.

Since κ is an uncountable cardinal in W and $\overline{M} \subseteq W$, it follows that κ is an uncountable cardinal in $\overline{M}$. Thus, although $\kappa = \omega_1^W$, it cannot be that $j(\kappa) = \omega_1^{\overline{M}}$, since κ itself is an uncountable cardinal strictly below $j(\kappa)$ in $\overline{M}$. Thus j is not elementary, and so the Łoś theorem fails for the ultrapower $\mathrm{Ult}(W, \mu^*)$. Specifically, the map j is not Σ_2-elementary, since the assertion "$\kappa = \omega_1$" has complexity Π_2. Even without any prior knowledge about the precise structure of the ultrapower $\mathrm{Ult}(W, \mu^*)$ and its transitive collapse $\overline{M}$, one can argue using the violation of collection that j is not Σ_2-elementary, and that, indeed, Łoś fails already for Σ_1-formulas. Recall that for all $\alpha < \kappa$, the model W has surjections $f : \omega \twoheadrightarrow \alpha$, but there cannot be a set in W collecting a family of such functions. Now to see that Łoś fails for Σ_1-formulas, observe that in the ultrapower there cannot exist a surjection from $[c_\omega] = \omega$ onto $[\mathrm{id}] = \kappa$, since otherwise, if $[g] : [c_\omega] \twoheadrightarrow [\mathrm{id}]$ were such a surjection, then $\{\alpha < \kappa \mid g(\alpha) \text{ is a surjection from } \omega \text{ onto } \alpha\}$ would be a set in μ^* (by the Łoś theorem for Δ_0-formulas), and so there would be a collecting set in W. If j were Σ_2-elementary, the ultrapower would have for all $\alpha < j(\kappa)$, a surjection from ω onto α, and hence a surjection from ω onto κ. □

Before continuing, let us remark on some subtle issues concerning the extent to which one can view the ultrapower of a ZFC- or even a ZFC$^-$ model as an internal construction inside such a model. One issue is that even in the case that a measure ν is definable in or perhaps even an element of a ZFC$^-$ model M, then although one can define in M the fundamental relations $=_\nu$ and $\in_\nu$ used to construct the ultrapower $\mathrm{Ult}(M, \nu)$, one seems unable in general to find representing sets in a definable way for the equivalence classes. Each equivalence class is, after all, a proper class in M, an issue usually resolved in the ZFC context by means of Scott's trick, where one restricts to the set of minimal-rank representatives in each equivalence class; but Scott's trick doesn't succeed in the ZFC- or ZFC$^-$ contexts, because the collection of minimal-rank representatives from a class may still not be a set, when one lacks the power set axiom. Thus, one seems to have difficulty performing the quotient operation, defining the ultrapower quotient structure as a first-order class model. Thus, if one wants to construct the ultrapower internally, it seems that one may be forced always to deal only with the pre-quotient structure, where one has only the equivalence relation $=_\nu$ rather than a true equality $=$ relation as in the quotient. A greater difficulty is that even if one should be able definably to find a representing set for each equivalence class, and thereby have a quotient representation of the ultrapower as a first-order class, one cannot necessarily perform the Mostowski collapse, even when the ultrapower is well-founded, because the $\in_\nu$ relation on those classes is not necessarily set-like in M. In fact, even a model of ZFC$^-$ containing a measurable cardinal is not always able to construct the ultrapower and take its Mostowski collapse. For an explicit example of this, suppose that κ is a measurable cardinal and fix an elementary embedding $j : V \to M$ by

a normal measure μ on κ and a strong limit cardinal λ of cofinality κ. It follows that $\lambda^+ < j(\lambda)$, since M is correct about ${}^\kappa\lambda$. Consider H_{λ^+}, which is a model of ZFC$^-$ containing μ and all functions $f : \kappa \to H_{\lambda^+}$. The collapsed ultrapower of H_{λ^+} by μ, therefore, is the same as the restriction $j : H_{\lambda^+} \to H^M_{j(\lambda)^+}$, which is not a subset of H_{λ^+} since $H^M_{j(\lambda)^+}$ contains ordinals above λ^+. Thus, even though H_{λ^+} sees that μ is a measure on κ, and is able to define the ultrapower relations and observe correctly that the ultrapower is well-founded, it is not able to perform the Mostowski collapse of this structure, since it lacks sufficient ordinals to do so. One way to describe the situation is that H_{λ^+} does not agree that a cardinal κ is measurable (in the sense of having a κ-complete nonprincipal ultrafilter on κ) if and only if there is an ultrapower embedding of the universe into a transitive class. So the equivalence of these two characterizations of measurability is not provable in ZFC$^-$.

Nevertheless, we can overcome these issues in the case of the model W and the measure μ^* that we construct in the proof of theorem 2.4. In particular, let us argue that both the Mostowski collapse $\overline{M}$ of $\mathrm{Ult}(W,\mu^*)$ and the corresponding ultrapower map $j : W \to \overline{M}$ of that proof are definable classes in W. As we observed in the proof, for any $\gamma < \kappa$, the Mostowski collapse of $[f]$ in $\mathrm{Ult}(W,\mu^*)$ is the same as the Mostowski collapse of $[f]$ in $\mathrm{Ult}(V[G_\gamma],\mu_\gamma)$. It follows that every function $g : \kappa \to V[G_\gamma]$ in W is equivalent on a set in μ to a function $g' : \kappa \to V[G_\gamma]$ with $g' \in V[G_\gamma]$. This means that we can compute in W the image of the Mostowski collapse of $[f]$ in $\mathrm{Ult}(W,\mu^*)$ by performing the collapse inside $V_\theta[G_\gamma]$ for large enough θ. Indeed, all that is needed is a sufficiently large transitive set A, such that $f \in A$ and for any function $g \in W$ having $g \in_{\mu^*} h \in A$ for some function h, then there is $g' \in A$ with $g =_{\mu^*} g'$. In this case, one shows that the Mostowski collapse of $[f]$ in $\mathrm{Ult}(W,\mu^*)$ is the same as the collapse of $[f]$ in $\langle A, \in_{\mu^*}\rangle/{=_{\mu^*}}$. Since there are abundant such A in W, such as $A = V_\theta[G_\gamma]$ for any sufficiently large θ, and they all give rise to the same value for the Mostowski collapse of $[f]$, it follows that in W we may definably associate any function f to its image under the Mostowski collapse of $\mathrm{Ult}(W,\mu^*)$. Thus, $\overline{M}$ is a definable class in W, and by considering the constant functions, we may also thereby define the ultrapower map $j : W \to \overline{M}$.

Let us now turn to the Lévy hierarchy of formulas in the language of set theory, where Σ_n-formulas and Π_n-formulas are defined as usual in a purely syntactical way. Recall that if a formula φ is obtained by bounded quantification over some Σ_n-formula, then there exists another Σ_n-formula φ' that is provably equivalent in ZFC to φ. The standard proof of this fact uses repeated applications of collection and the pairing axiom, and so it follows that the equivalence between φ and φ' can also be proved in ZFC$^-$. Corollary 9 shows that the use of collection is essential to this argument, and one cannot obtain in some other way a formula φ' of reduced complexity that is provably equivalent in ZFC- to φ.

Corollary 9. *Relative to a measurable cardinal, the collection of formulas that are provably equivalent in* ZFC- *to a* Σ_1*-formula or a* Π_1*-formula is not closed under bounded quantification.*

Proof. Let $\varphi(x)$ be the formula asserting that all elements of x are countable, meaning that for each nonempty $y \in x$ there exists a surjection from ω onto y. The formula $\varphi(x)$ is clearly obtained by bounded universal quantification over a

Σ_1-formula, but we will show that it is not provably equivalent in ZFC- to any Σ_1-formula or Π_1-formula. Consider the Σ_1-elementary embedding $j : W \to \overline{M}$ from theorem 8 and note that the model W satisfies $\varphi(\kappa)$ since it has collapsing functions for all ordinals below κ. If there would be some Σ_1-formula $\varphi'(x)$ such that ZFC- proves $\forall x(\varphi(x) \leftrightarrow \varphi'(x))$, then it would follow that $W \models \varphi'(\kappa)$, and so $\overline{M} \models \varphi'(j(\kappa))$ by the Σ_1-elementarity of $j : W \to \overline{M}$. Since $\overline{M} \models$ ZFC- , it would then follow that $M \models \varphi(j(\kappa))$, which means that κ would be countable in M, which is not the case. The same argument shows that there cannot be a Π_1-formula provably equivalent in ZFC- to φ. □

We will improve this result by avoiding the need for a measurable cardinal in section §2.7. Also, in subsequent sections, we will obtain counterexamples to the Łoś theorem involving ultrafilters that exist inside the model, including ultrafilters on ω when $P(\omega)$ exists, as well as other violations of the Łoś and Gaifman theorems, and other counterexamples that do not require any large cardinals.

2.5. **A cofinal restriction of an elementary embedding.** In the previous section we proved that the Gaifman theorem can fail for ZFC- models, in the sense that there can be $j : M \to N$ for transitive ZFC- models M and N, which is Σ_1-elementary and cofinal, but not Σ_2-elementary. We would like now to describe a dual situation, where one has a fully elementary but non-cofinal embedding $j : M \to N$ of transitive ZFC- models, whose canonical cofinal restriction $j : M \to \bigcup j\,"\,M$ is not elementary. Note that since M and N are transitive, then so is $\bigcup j\,"\,M$, since it is a union of transitive sets (using replacement, one shows that M satisfies for every set, there is a set that is its transitive closure). The restriction $j : M \to \bigcup j\,"\,M$ is a Σ_1-elementary embedding, since it is clearly Δ_0-elementary and also cofinal, but theorem 10 shows that it need not be Σ_2-elementary if collection fails in M. This stands in contrast to the situation when $M \models \mathrm{ZFC}^-$ and $j : M \to N$ is fully elementary, since one can then argue that $\bigcup j\,"\,M \prec N$ by the Tarski-Vaught test and conclude that $j : M \to \bigcup j\,"\,M$ is fully elementary.

Theorem 10. *Relative to a measurable cardinal, it is consistent that there are transitive models M and N of* ZFC- *with a fully elementary embedding $j : M \to N$, whose cofinal restriction $j : M \to \bigcup j\,"\,M$ is an embedding of* ZFC- *models that is not Σ_2-elementary.*

Proof. Let us suppose that κ is a measurable cardinal in V. Let $V[G]$ be the Lévy collapse of κ, meaning that $G \subseteq \mathbb{P} = \mathrm{Coll}(\omega, {<}\kappa)$ is V-generic, and let $W = \bigcup_{\gamma<\kappa} V[G_\gamma]$, as in section §2.3, so that we know $W \models$ ZFC-. Let $j : V \to M$ be the ultrapower by a normal measure μ on κ in V, and consider the forcing $j(\mathbb{P}) = \mathrm{Coll}(\omega, {<}j(\kappa))$, which can be factored as $j(\mathbb{P}) \cong \mathbb{P} \times \mathbb{P}_{\kappa,j(\kappa)}$, where $\mathbb{P}_{\kappa,j(\kappa)}$ is the part of the forcing collapsing cardinals in the interval $[\kappa, j(\kappa))$. Suppose that $H \subseteq \mathbb{P}_{\kappa,j(\kappa)}$ is $V[G]$-generic. Since $G \times H$ is V-generic for $j(\mathbb{P})$ and $j\,"\,G = G$, it follows that j lifts to the fully elementary $j : V[G] \to M[j(G)]$ in $V[G][H]$, where $j(G) = G \times H$. Since W is definable in $V[G]$, using Laver's result [Lav07] on the uniform definability of the ground model in the forcing extension from the generic filter and a poset-dependent parameter, we may restrict j to W and obtain in $V[G][H]$ the fully elementary embedding $j \restriction W : W \to j(W)$, where $j(W) = \bigcup_{\gamma<j(\kappa)} M[j(G)_\gamma]$ by the uniformity of Laver's definition. The cofinal restriction $j \restriction W : W \to \bigcup j\,"\,W$ is Σ_1-elementary, but it is not fully elementary, since all ordinals below κ are countable

in W, but the ordinal κ below $j(\kappa)$ cannot be countable in $\bigcup j\,``\,W$, since it is easy to see that $\bigcup j\,``\,W \subseteq \bigcup_{\gamma<\kappa} M[j(G)_\gamma] = \bigcup_{\gamma<\kappa} M[G_\gamma]$ and no $M[G_\gamma]$ can collapse cardinals above γ. The assertion "$\kappa = \omega_1$" has complexity Π_2, and we observe more precisely that $j : W \to \bigcup j\,``\,W$ is not $\Sigma_2$-elementary. The embedding $j \restriction W : W \to j(W)$ is thus not cofinal, but to see this directly, note that, for example, the reals added by $j(G)_\kappa$ are in $j(W)$ but not in $\bigcup j\,``\,W$.

Indeed, $\bigcup j\,``\,W = \bigcup_{\gamma<\kappa} M[G_\gamma]$, since $x \in M[G_\gamma]$ implies that $x = j(f)(\kappa)$ for some function $f \in V[G_\gamma]$, as the restriction of j to $V[G_\gamma]$ is a lift of the ultrapower map j and hence an ultrapower map as well, and consequently $x \in j(\operatorname{ran} f) \subseteq \bigcup j\,``\,W$. It follows that $\bigcup j\,``\,W$ satisfies ZFC-. Note finally that the map $j \restriction W : W \to \bigcup j\,``\,W$ is precisely the ultrapower of W by μ^* as described in section §2.4 by uniqueness of the lift of j to each $V[G_\gamma]$. $\square$

Theorem 10 can be used to show that standard arguments from seed theory[9], permissible in the context of ZFC^- models, can fail in the context of ZFC- models.

Corollary 11. *Relative to a measurable cardinal, it is consistent that there are elementary embeddings $j : M \to N$ of transitive models of* ZFC- *and sets $S \subseteq \bigcup j\,``\,M$ such that the seed hull $\mathbb{X}_S = \{j(f)(s) \mid s \in [S]^{<\omega}, f \in M\}$ is not a Σ_1-elementary substructure of N, and such that the restriction $j : M \to \mathbb{X}_S$ is not Σ_2-elementary.*

Proof. Consider the elementary embedding $j \restriction W : W \to j(W)$ of the proof of theorem 10 and let $X_{\{\kappa\}} = \{j(f)(\kappa) \mid f : \kappa \to W, f \in W\}$ be the seed hull of $\{\kappa\}$ via j. Observe that $X_{\{\kappa\}} = \bigcup_{\gamma<\kappa} M[G_\gamma]$. The seed hull $X_{\{\kappa\}}$ is not a Σ_1-elementary substructure of $j(W)$, since if it were, then $\mathbb{X}_{\{\kappa\}}$ would have a surjection from ω onto κ, but we know that this is not the case, since no $M[G_\gamma]$ collapses κ if $\gamma < \kappa$. $\square$

2.6. **Violating Łoś with a measurable cardinal inside the model.** We shall now modify the construction of section §2.4 in order to arrive at a more definitive violation of the Łoś theorem for ZFC- models, by producing a model $W \models$ ZFC- in which there is a measurable cardinal κ whose powerset $P(\kappa)$ exists in W and for which there is a κ-complete normal ultrafilter μ on κ in W, whose ultrapower $\mathrm{Ult}(W,\mu)$ is well-founded and can be constructed and collapsed inside W, but the ultrapower map $j : W \to \mathrm{Ult}(W,\mu)$ is not elementary.

Theorem 12. *Relative to a measurable cardinal, it is consistent with* ZFC- *that there is a measurable cardinal κ for which $P(\kappa)$ exists and there is a κ-complete normal measure on κ, whose ultrapower is a well-founded model of* ZFC- *that can be constructed and collapsed, but for which the Łoś theorem fails at the Σ_1-level and the ultrapower map is not Σ_2-elementary.*

Proof. The idea is to combine the methods of sections §2.1 and §2.4, but performing all the collapse forcing only above the measurable cardinal, so that this cardinal and its measure will be preserved to the extension and to the resulting model W of ZFC-. Suppose that κ is a measurable cardinal in $V \models \mathrm{ZFC} + \mathrm{GCH}$, and let $\lambda = \kappa^{(+)\kappa} = \aleph_{\kappa+\kappa}$, the κ^{th} cardinal successor to κ. (The GCH assumption can be relaxed simply by using $\beth_{\kappa+\kappa}$ in place of λ.) Let $\mathbb{P}$ be the Lévy collapse of λ meaning the bounded-support product $\mathrm{Coll}(\kappa^+, {<}\lambda) = \Pi_{\alpha<\kappa} \mathrm{Coll}(\kappa^+, \aleph_{\kappa+\alpha})$. Although the forcing $\mathrm{Coll}(\kappa^+, \aleph_{\kappa+\alpha})$ on any coordinate $\aleph_{\kappa+\alpha}$ is $\leq\kappa$-closed, the $\mathbb{P}$

[9] For a review of elementary seed theory, see for instance [Ham97].

forcing is not $\leq\kappa$-closed because of the bounded-support requirement, and in the limit $\mathbb{P}$ will actually collapse λ to κ, since the function mapping $\alpha<\kappa$ to the first ordinal used on coordinate $\aleph_{\kappa+\alpha}$ will map κ onto λ. But every initial segment of the forcing $\mathbb{P}_\gamma=\Pi_{\alpha\leq\gamma}\operatorname{Coll}(\kappa^+,\aleph_{\kappa+\alpha})$, for $\gamma<\kappa$, is $\leq\kappa$-closed and therefore adds no subsets to κ.

Suppose that $G\subseteq\mathbb{P}$ is V-generic for this forcing, and let $G_\gamma=G\cap\mathbb{P}_\gamma$. Our intended model is $W=\bigcup_{\gamma<\kappa}V[G_\gamma]$, which satisfies ZFC-, by arguments analogous to those previously given, but not collection. To see that collection fails in W, observe that for each $\alpha<\kappa$ there exists a surjective function $f:\kappa^+\twoheadrightarrow\aleph^V_{\kappa+\alpha}$ in W, but there cannot be a set in W collecting a family of such functions.

Since all $\mathbb{P}_\gamma$ are $\leq\kappa$-closed, we have that $P(\kappa)^V=P(\kappa)^{V[G_\gamma]}=P(\kappa)^W$. Thus, $P(\kappa)$ exists unchanged in W. Furthermore, if μ is any normal measure on κ in V, then μ continues to be a normal measure on κ in W. Note that for every $x\in W$, the collection of all $f:\kappa\to x$ forms a set in W since if $x\in V[G_\gamma]$, then no new functions from κ to x are added by subsequent collapses. Therefore W can construct the ultrapower $\operatorname{Ult}(W,\mu)$ as the union of the ultrapowers $\operatorname{Ult}(x,\mu)$ over all transitive sets $x\in W$ and take its Mostowski collapse (this requires replacement) to a model $\overline{M}$ and let $j:W\to\overline{M}$ be the resulting embedding.

We shall now observe that Łoś already fails at the Σ_1-level for this ultrapower and that the map $j:W\to\overline{M}$ is not Σ_2-elementary, using the violation of collection described above. Let h be a function on κ in $V\subseteq W$ such that $h(\alpha)=\aleph^V_{\kappa+\alpha}$, and recall that for all $\alpha<\kappa$, the model W has surjections from κ^+ onto $h(\alpha)$ but no collecting set of such functions. It follows that there cannot be a surjection $[f]:j(\kappa^+)\twoheadrightarrow[h]$ in the ultrapower, since otherwise the set $\{\alpha<\kappa\mid f:\kappa^+\twoheadrightarrow h(\alpha)\}$ would be an element of μ (by Łoś for Δ_0-formulas), yielding a collecting set, and thus Łoś fails at the Σ_1-level. It also follows that the ultrapower map j is not elementary for the Π_2-formula $\forall\alpha<\kappa\exists f\,f:\kappa^+\twoheadrightarrow h(\alpha)$ using κ^+ and h as parameters, since otherwise the ultrapower would have a surjection from κ^+ onto $j(h)(\kappa)=[h]$.

It remains to argue that $\overline{M}$ is nevertheless a model of ZFC-. Let $j:V\to M$ be the ultrapower embedding by μ in V. Since the forcing $\mathbb{P}_\gamma$ is $\leq\kappa$-closed, it follows that j lifts uniquely to $j_\gamma:V[G_\gamma]\to M[j(G_\gamma)]$, where $j(G_\gamma)$ is the filter in $j(\mathbb{P}_\gamma)$ generated by $j\,"\,G_\gamma$. (This is generic since every open dense set $D\subseteq j(\mathbb{P}_\gamma)$ is $j(\vec{D})(\kappa)$ for some $\vec{D}=\langle D_\alpha\mid\alpha<\kappa\rangle$ with $D_\alpha\subseteq\mathbb{P}_\gamma$ open dense, and by $\leq\kappa$-closure it follows that $\overline{D}=\bigcap_\alpha D_\alpha$ remains dense and has $j(\overline{D})\subseteq D$; since G_γ meets $\overline{D}$, it follows that $j\,"\,G_\gamma$ meets $j(\overline{D})$ and hence D, as desired.) We claim that $\overline{M}=\bigcup_{\gamma<\lambda}M[j(G_\gamma)]$ and $j=\bigcup_{\gamma<\kappa}j_\gamma$ by the map that associates $[f]_\mu\in\operatorname{Ult}(W,\mu)$ to $[f]_\mu$ in any $\operatorname{Ult}(V[G_\gamma],\mu)$ for which $f\in V[G_\gamma]$. This association is well-defined and $\in$-preserving, since if $f\in V[G_\gamma]$ and $\gamma<\delta<\kappa$, then $j_\gamma(f)(\kappa)=[f]_\mu$ as computed in $\operatorname{Ult}(V[G_\gamma],\mu)$, which is the same as $j_\delta(f)(\kappa)=[f]_\mu$ as computed in $\operatorname{Ult}(V[G_\delta],\mu)$, by the uniqueness of the lifted embeddings. The association is onto and hence an isomorphism, and it commutes with the ultrapower maps. Since $\overline{M}=\bigcup_{\gamma<\lambda}M[j(G_\gamma)]$, it follows that $\overline{M}$ is obtained from M by Lévy collapsing all cardinals below $\aleph_{j(\kappa)+\kappa}$ to $j(\kappa^+)$ using bounded-support, as computed in M, and it follows by the usual arguments that $\overline{M}\models$ ZFC-. $\square$

2.7. **Violating the Łoś and Gaifman theorems without large cardinals.** Analogous arguments as in theorems 8, 10 and 12 and corollaries 9 and 11 work in the context of elementary embeddings characterizing smaller large cardinals. For

instance, if κ is weakly compact, we can find a transitive set $M \models$ ZFC of size κ with $\kappa, V_\kappa \in M$ and an ultrapower map $j : M \to N$ by an M-normal measure on κ such that N is transitive. The poset $\mathrm{Coll}(\omega, {<}\kappa)$ is an element of M as it is definable over V_κ, and so in $V[G]$, the Lévy collapse of κ, we may form the forcing extension $M[G]$ and let $W = \bigcup_{\gamma<\kappa} M[G_\gamma]$, which satisfies ZFC-, but not collection. The rest of the constructions now proceed identically to those previously given.

In fact, these arguments can be modified to work with elementary embeddings of transitive models whose existence follows directly from ZFC, thereby avoiding any need for large cardinals.

Theorem 13. *There are transitive set models M and N of* ZFC- *with a fully elementary embedding $j : M \to N$, whose cofinal restriction $j : M \to \bigcup j\,"\,M$ is not Σ_2-elementary.*

Proof. We start by constructing an appropriate elementary embedding $j : M \to N$ with critical point ω_1^M. First, we may assume without loss of generality that GCH holds, by passing if necessary to an inner model. Let $\theta > 2^{\aleph_{\omega_1}}$ be any regular cardinal. Let X be a countable elementary substructure of H_θ with Mostowski collapse $\pi_X : X \to M$, and observe that ω_1^M exists in M, and that $M \models \mathrm{ZFC}^-$. Using CH, let $Y \supseteq X$ be an elementary substructure of H_θ of size ω_1 with $Y^\omega \subseteq Y$. Let $\pi_Y : Y \to N$ be the Mostowski collapse of Y, and observe that $N \models \mathrm{ZFC}^-$ and $N^\omega \subseteq N$. The composition map $j = \pi_Y \circ \pi_X^{-1}$ is then an elementary embedding $j : M \to N$ with critical point ω_1^M, that is $\omega_1^M < j(\omega_1^M) = \omega_1^N = \omega_1$, so that ω_1^M is a countable ordinal in N.

Working inside of the model M, note that $P(\aleph_{\omega_1})$ exists and we may thus consider the Lévy collapse of $\aleph_{\omega_1}$ to ω_2, meaning the bounded-support product $\mathbb{P} = \mathrm{Coll}(\omega_2, {<}\aleph_{\omega_1}) = \Pi_{\beta<\omega_1} \mathrm{Coll}(\omega_1, \aleph_\beta)$. The forcing $\mathbb{P}$ is countably closed in M, but not $\leq\omega_1$-closed because of the bounded-support requirement, and in the limit $\mathbb{P}$ will actually collapse $\aleph_{\omega_1}$ to ω_1, since the function mapping $\alpha < \omega_1$ to the first ordinal mentioned on coordinate $\aleph_\alpha$ will map ω_1 onto $\aleph_{\omega_1}$.

Since $\mathbb{P}$ is the Lévy collapse of $\aleph_{\omega_1}^M$ to ω_2^M, as computed in the model M, it follows by elementarity that $j(\mathbb{P}) = \mathrm{Coll}(\omega_2, {<}\aleph_{\omega_1})^N$ is the Lévy collapse of $\aleph_{\omega_1}^N$ to ω_2^N, as computed in N. Note that $j(\mathbb{P})$ is countably closed, since it is countably closed in N and $N^\omega \subseteq N$. We aim to lift the elementary embedding $j : M \to N$ to forcing extensions of M and N by $\mathbb{P}$ and $j(\mathbb{P})$, respectively, using generic filters that exist in V. Since M is countable, there exists an M-generic filter $G \subseteq \mathbb{P}$ that is generated by a countable descending sequence $\{p_n \mid n \in \omega\}$ of conditions in $\mathbb{P}$. By closure of $j(\mathbb{P})$, we can find a condition $q \in j(\mathbb{P})$ that is below all the $j(p_n)$, and since N has size ω_1, we can build an N-generic filter $j(G) \subseteq j(\mathbb{P})$ having q as an element. It follows that $j\,"\,G \subseteq j(G)$, and so we can lift the embedding in V to $j : M[G] \to N[j(G)]$.

For each $\gamma < \omega_1^M$, let $G_\gamma = G \cap \mathbb{P}_\gamma$. Our intended model is $W = \bigcup_{\gamma<\omega_1^M} M[G_\gamma]$, which satisfies ZFC-, by arguments analogous to those previously given, but not collection. To see that collection fails in W, observe that for every $\alpha < \aleph_{\omega_1}^M$, the model W has a surjective function $f : \omega_2^M \twoheadrightarrow \alpha$, but no set containing a family of such functions by the chain condition of $\mathbb{P}_\gamma$ as the GCH holds in M, and thus there cannot be a set in W collecting a family of such functions.

We may assume without loss of generality that W is a definable class in $M[G]$, by starting the construction if necessary[10] in an inner model such as L. We may therefore restrict j to W and obtain, as in theorem 10, a fully elementary embedding $j \restriction W : W \to j(W)$, where $j(W) = \bigcup_{\gamma<\omega_1} N[j(G)_\gamma)]$. The canonical cofinal restriction $j \restriction W : W \to \bigcup j " W$ is Σ_1-elementary, but it is not fully elementary, since for each $\alpha < \aleph^M_{\omega_1}$ there is a function in W collapsing α to ω^M_2, but there is no function in $\bigcup j " W \subseteq \bigcup_{\gamma<\omega^M_1} N[j(G)_\gamma]$ that collapses the cardinal $\aleph^N_{\omega^M_1}$ which is strictly below $j(\aleph^M_{\omega_1}) = \aleph^N_{\omega_1}$, since no $N[j(G)_\gamma]$ collapses cardinals above $\aleph^N_\gamma$. Altogether, we have observed that there are no cardinals between ω^M_2 and $\aleph^M_{\omega_1}$ in W, but there are cardinals in $\bigcup j " W$ between $j(\omega^M_2) = \omega^N_2$ and $j(\aleph^M_{\omega_1}) = \aleph^N_{\omega_1}$, since $\aleph^N_{\omega^M_1}$ itself is such a cardinal. Since the assertion "$\lambda = \delta^+$" where $\lambda = \aleph^M_{\omega_1}$ and $\delta = \omega^M_2$ has complexity Π_2, it follows that the cofinal restriction $j \restriction W : W \to \bigcup j " W$ is not Σ_2-elementary. $\square$

Using the map $j : W \to \bigcup j " W$ from theorem 13, it is easy to obtain the result of corollary 9, this time without large cardinals.

Theorem 13 can also be used to provide failures of the elementarity of seed hulls, but in addition to the violations of corollary 11, we now obtain for a single elementary embedding $j : M \to N$ uncountably many distinct seed hulls $\mathbb{X}_S$ that are not elementary substructures of N and that are not generated by a single seed. Indeed, consider the elementary embedding $j \restriction W : W \to j(W)$ from the proof of theorem 13. Let $S \subseteq \bigcup j " W$ be any seed set containing $\aleph^N_{\omega^M_1}$ as a subset and note that $\mathbb{X}_S \subseteq \bigcup j " W$. The seed hull $\mathbb{X}_S$ cannot be generated by a single seed, since W is countable, but $\mathbb{X}_S$ is uncountable. If $\mathbb{X}_S$ were Σ_1-elementary in $j(W)$, then $\mathbb{X}_S$ and hence $\bigcup j " W$ would contain a surjection from ω^M_2 onto $\aleph^N_{\omega^M_1}$, but we argued already in theorem 13 that this is not the case. Observe that there are uncountably many distinct seed hulls: Since N is closed under countable sequences, it follows that $N[j(G)]^\omega \subseteq N[j(G)]$ and consequently $j(W)^\omega \subseteq j(W)$, and so $j \restriction W$ is an element of $j(W)$. The model $j(W)$ can construct seed hulls for seed sets $S \in j(W)$ of different cardinalities in $j(W)$, and the corresponding seed hulls $\mathbb{X}_S$ are thus all distinct. Lastly, the map $j : W \to \mathbb{X}_S$ is a Σ_1-elementary cofinal map that is not Σ_2-elementary, and the same is true for the map $\pi \circ j : W \to \operatorname{ran}(\pi)$ where $\pi : \mathbb{X}_S \to \operatorname{ran}(\pi)$ denotes the transitive collapse of $\mathbb{X}_S$. These two maps provide therefore counterexamples to the Gaifman theorem in the ZFC- context for maps that are not ultrapower maps.

Essentially the same arguments as in theorem 13 allow us to obtain a failure of the Łoś theorem, without any need for large cardinals.

Theorem 14. *There is a transitive set $M \models$ ZFC- and an M-normal measure $\mu \subseteq P(\omega_1)^M$ for which the ultrapower of M by μ is a well-founded model of* ZFC-*, but the ultrapower map is not Σ_2-elementary, and Łoś fails at the Σ_1-level.*

Proof. Following the proof of theorem 13, we obtain an elementary embedding $j : M \to N$ with critical point ω^M_1, where M is the Mostowski collapse of some countable $X \prec H_\theta$ for a regular cardinal $\theta > 2^{\aleph_{\omega_1}}$ and N is the Mostowski collapse of some $Y \prec H_\theta$ with $X \subseteq Y$. If $\mu \subseteq P(\omega_1)^M$ is the M-normal measure on ω^M_1 that

[10] In general, it seems unclear whether W is definable in $M[G]$, since the proof of Laver's theorem [Lav07] on the definability of the ground model makes essential use of the power set axiom.

is obtained from $j : M \to N$ by using ω_1^M as a seed, then the ultrapower $\mathrm{Ult}(M, \mu)$ of M, using functions on ω_1^M in M, is well-founded, and so we may assume without loss of generality that $j : M \to N$ is the ultrapower by μ. It follows, in particular, that N is countable.

Again, we assume that GCH holds. As in theorem 13, we consider the bounded-support product $\mathbb{P} = \mathrm{Coll}(\omega_2, {<}\aleph_{\omega_1})^M$, and since M is countable, choose in V an M-generic filter $G \subseteq \mathbb{P}$ and let $G_\gamma = G \cap P_\gamma$ for each $\gamma < \omega_1^M$. Our intended model is again $W = \bigcup_{\gamma<\omega_1^M} M[G_\gamma]$. Since each initial forcing $\mathbb{P}_\gamma$ is $\leq\omega_1$-closed in M, it follows, as in theorem 12, that $j : M \to N$ lifts uniquely to $j_\gamma : M[G_\gamma] \to N[j(G_\gamma)]$, where $j(G_\gamma)$ is the N-generic filter in $j(\mathbb{P})$ generated by j " G_γ, and the map j_γ is necessarily equal to the ultrapower map of $M[G_\gamma]$ by μ. It follows that the ultrapower $\mathrm{Ult}(W, \mu)$ of W, using functions on ω_1^M in W, is well-founded, since if $j : W \to \overline{N}$ is the ultrapower map, then $\overline{N} = \bigcup_{\gamma<\omega_1^M} N[j(G_\gamma)]$ and $j = \bigcup_{\gamma<\omega_1^M} j_\gamma$ via the isomorphism that associates $[f]_\mu \in \mathrm{Ult}(W, \mu)$ to $[f]_\mu$ in any $\mathrm{Ult}(M[G_\gamma], \mu)$ for which $f \in M[G_\gamma]$. Thus, $\overline{N}$ satisfies ZFC−. An argument analogous to that in the proof of theorem 13 uses the violation of collection to show that Łoś fails at the Σ_1-level and the ultrapower map $j : W \to \overline{N}$ is not Σ_2-elementary. Thus, j witnesses the failure of both the Łoś and Gaifman theorems. □

2.8. **Violating Łoś for ultrapowers on ω.** In this final section, we aim to produce a counterexample to the Łoś and Gaifman theorems for ZFC− models for ultrapowers by ultrafilters on ω.

Theorem 15. *In a forcing extension $V[G]$ by the Lévy collapse $G \subseteq \mathrm{Coll}(\aleph_1, {<}\aleph_\omega)$ there is a transitive class inner model $W \models$ ZFC− in which $P(\omega)$ exists and there is an ultrafilter μ on ω, such that the ultrapower map $j : W \to \mathrm{Ult}(W, \mu)$ is definable in W, but is not Σ_2-elementary and, indeed, Łoś fails at the Σ_1-level.*

Proof. First, we may assume without loss of generality that GCH holds in V, by passing if necessary to an inner model such as L. Suppose that $G \subseteq \mathrm{Coll}(\aleph_1, {<}\aleph_\omega)$ is V-generic for the Lévy collapse up to $\aleph_\omega$, that is, the finite-support product $\mathbb{P} = \Pi_{n<\omega} \mathrm{Coll}(\aleph_1, \aleph_n)$. Let $W = \bigcup_n V[G_n]$, where $G_n = G \cap \mathbb{P}_n$ and $\mathbb{P}_n = \Pi_{k\leq n} \mathrm{Coll}(\aleph_1, \aleph_k)$. The model W satisfies ZFC−, but not collection. Since each $\mathbb{P}_n$ is countably closed and therefore does not add subsets to ω, we have $P(\omega)^V = P(\omega)^{V[G_n]} = P(\omega)^W$, meaning that $P(\omega)$ exists in W. If μ is any nonprincipal ultrafilter on ω in V, then it continues to have this property in W. Let $j : W \to \mathrm{Ult}(W, \mu)$ be the ultrapower of W by μ, as computed by W. The model W is able to take the ultrapower since for every $x \in W$, the collection of all $f : \omega \to x$ forms a set in W since if $x \in V[G_n]$, then no new functions from κ to x are added by the subsequent collapses. The failure of Łoś at the Σ_1-level and the failure of elementarity for the ultrapower map at the Σ_2-level follow from the violation of collection as in the previous arguments. The Gaifman theorem fails for this embedding as well, as it is Σ_1-elementary and cofinal. □

3. Some final remarks

It is clear that the method of proof of our theorems is both flexible and general and would easily adapt to the use of other cardinals than the ones we have used. For example, in section 2.1 we could have collapsed to different cardinals, as in theorems 12 and 15, in order to show that other successor cardinals can be singular, or that

several cardinals might be singular. Indeed, [Zar96] provides a general framework for producing models of ZFC-, involving forcing over the weak product of ω-many copies of a family of forcing notions, and forming the desired model via increasing finite portions of the product. The proof that his general framework succeeds amounts essentially to the particular arguments we made for our iterations, where we swapped two copies of the forcing $\mathbb{P}_n$ or $\mathbb{P}_\gamma$ and appealed to almost homogeneity. Although we could have appealed to Zarach's general framework, we chose simply to give a direct argument in each case in order to achieve a self-contained presentation. But we refer any reader interested in producing even more badly behaved models of ZFC- to consult Zarach's general framework. The violations of Łoś that we produced in our example can also be summarized in a general framework, as may be deduced from the examples we provided in the paper. For our own part, we shall from now on prefer the models of ZFC^- over ZFC-, and we take the results of this article to show definitively that ZFC- is the wrong theory for most applications of set theory without power set.

References

[Abr10] U. Abraham. Proper Forcing. In *Handbook of Set Theory*, pages 333–394. Springer, New York, 2010.

[Dev84] Keith J. Devlin. *Constructibility.* Perspectives in Mathematical Logic. Springer-Verlag, Berlin, 1984.

[Gai74] H. Gaifman. Elementary embeddings of models of set-theory and certain subtheories. In *Axiomatic set theory (Proc. Sympos. Pure Math., Vol. XIII, Part II, Univ. California, Los Angeles, Calif., 1967)*, pages 33–101. Amer. Math. Soc., Providence R.I., 1974.

[Git11] V. Gitman. Ramsey-like cardinals. *The Journal of Symbolic Logic*, 76(2):519–540, 2011.

[Ham97] Joel David Hamkins. Canonical seeds and Prikry trees. *J. Symbolic Logic*, 62(2):373–396, 1997.

[HJ10] J. D. Hamkins and T. A. Johnstone. Indestructible strong unfoldability. *Notre Dame J. Form. Log.*, 51(3):291–321, 2010.

[Jec66] T. Jech. Interdependence of weakened forms of the axiom of choice. *Comment. Math. Univ. Carolinae*, 7:359–371, 1966.

[Jec03] T. Jech. *Set Theory.* Springer Monographs in Mathematics. Springer–Verlag, New York, third edition, 2003.

[Kan03] A. Kanamori. *The Higher Infinite.* Springer Monographs in Mathematics. Springer–Verlag, New York, second edition, 2003.

[Lav07] Richard Laver. Certain very large cardinals are not created in small forcing extensions. *Annals of Pure and Applied Logic*, 149(1–3):1–6, 2007.

[Lév64] A. Lévy. The interdependence of certain consequences of the axiom of choice. *Fund. Math.*, 54:135–157, 1964.

[LS67] Azriel Lévy and Robert M. Solovay. Measurable cardinals and the Continuum Hypothesis. *Israel Journal of Mathematics*, 5:234–248, 1967.

[Nee10] I. Neeman. Determinacy in $L(\mathbb{R})$. In *Handbook of Set Theory*, pages 1877–1950. Springer, New York, 2010.

[Sol70] Robert M. Solovay. A model of set-theory in which every set of reals is Lebesgue measurable. *Annals of Mathematics. Second Series*, 92(1):1–56, 1970.

[Zar82] Andrzej Zarach. Unions of ZF^--models which are themselves ZF^--models. In *Logic Colloquium '80 (Prague, 1980)*, volume 108 of *Stud. Logic Foundations Math.*, pages 315–342. North-Holland, Amsterdam, 1982.

[Zar96] Andrzej M. Zarach. Replacement $\nrightarrow$ collection. In *Gödel '96 (Brno, 1996)*, volume 6 of *Lecture Notes Logic*, pages 307–322. Springer, Berlin, 1996.

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